System of particles and rotational motion

A. CENTER OF MASS

Definition & Position Vector

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M}$$

Discrete System

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{M}; \quad M = \sum m_i$$

Two-Particle System

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$m_1 d_1 = m_2 d_2$ (COM divides in inverse ratio)

Continuous Distribution

$$x_{cm} = \frac{\int x\, dm}{\int dm}; \quad dm = \lambda\, dl \text{ or } \sigma\, dA \text{ or } \rho\, dV$$

Velocity & Acceleration of COM

$$\vec{v}_{cm} = \frac{\sum m_i \vec{v}_i}{M}; \quad \vec{a}_{cm} = \frac{\sum m_i \vec{a}_i}{M}$$

$M\vec{a}_{cm} = \vec{F}_{ext}$ (Newton's 2nd for system)

Conservation of Linear Momentum

$$\vec{F}_{ext} = 0 \;\Rightarrow\; \vec{p}_{total} = \text{const}$$

Standard COM Results

Body	COM from base
Semicircular arc	$\frac{2R}{\pi}$
Semicircular disc	$\frac{4R}{3\pi}$
Solid hemisphere	$\frac{3R}{8}$
Hollow hemisphere	$\frac{R}{2}$
Solid cone (h)	$\frac{h}{4}$
Hollow cone (h)	$\frac{h}{3}$

B. MOMENTUM & COLLISIONS

Linear Momentum & Impulse

$\vec{p} = m\vec{v}$; $\quad \vec{J} = \Delta\vec{p} = \vec{F}\Delta t$

Coefficient of Restitution

$$e = \frac{v_2 - v_1}{u_1 - u_2} \qquad 0 \le e \le 1$$

$e = 1$: elastic; $\quad e = 0$: perfectly inelastic

Elastic Collision (1D)

$$v_1 = \frac{(m_1 - m_2)u_1 + 2m_2 u_2}{m_1 + m_2}$$

$$v_2 = \frac{(m_2 - m_1)u_2 + 2m_1 u_1}{m_1 + m_2}$$

Equal masses: velocities exchange completely.

Inelastic Collision

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \quad \text{(perfectly inelastic)}$$

$$\Delta KE = \frac{m_1 m_2 (u_1 - u_2)^2}{2(m_1 + m_2)} \quad \text{(lost)}$$

Explosion Process

No external force $\Rightarrow$ momentum conserved

$m\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 = \text{const}$

MOST IMPORTANT

- COM of system doesn't move if $F_{ext} = 0$
- Loss in KE in perfectly inelastic:

$$\Delta KE = \frac{1}{2}\mu(u_1 - u_2)^2; \; \mu = \frac{m_1 m_2}{m_1 + m_2}$$

C. ROTATIONAL KINEMATICS

Angular Quantities

θ = angular displacement (rad)

$\omega = \frac{d\theta}{dt}$ = angular velocity (rad/s)

$\alpha = \frac{d\omega}{dt}$ = angular acceleration (rad/s²)

Kinematic Equations (Const. α)

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta$$
$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t$$

Linear ↔ Angular Relations

$$v = r\omega; \quad a_t = r\alpha; \quad a_c = \omega^2 r = \frac{v^2}{r}$$

D. TORQUE & ANGULAR MOMENTUM

Torque

$$\vec{\tau} = \vec{r} \times \vec{F}; \quad |\tau| = rF\sin\theta$$

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}; \quad |\vec{L}| = mvr\sin\theta$$

Torque–Angular Momentum Relation

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum

$$\vec{\tau}_{ext} = 0 \;\Rightarrow\; \vec{L} = I\omega = \text{const}$$

Diver tucks in: $I\downarrow \Rightarrow \omega\uparrow$

E. MOMENT OF INERTIA

Definition & Radius of Gyration

$$I = \sum m_i r_i^2 = \int r^2\, dm; \quad I = MK^2$$

K = radius of gyration

Theorems

Parallel Axis: $I = I_{cm} + Md^2$

Perpendicular Axis (lamina): $I_z = I_x + I_y$

Standard MOI Values

Body	Axis	I
Ring (R)	diameter	$\frac{MR^2}{2}$
Ring (R)	central	MR^2
Disc (R)	central	$\frac{MR^2}{2}$
Disc (R)	diameter	$\frac{MR^2}{4}$
Solid sphere	diameter	$\frac{2MR^2}{5}$
Hollow sphere	diameter	$\frac{2MR^2}{3}$
Rod (L)	centre ⊥	$\frac{ML^2}{12}$
Rod (L)	end ⊥	$\frac{ML^2}{3}$
Solid cylinder	axis	$\frac{MR^2}{2}$
Hollow cylinder	axis	MR^2
Rect. plate	through C	$\frac{M(a^2+b^2)}{12}$

Memory Tip

Ring > **Hollow sphere** > **Solid sphere** > **Disc**
(for same mass and radius, about diameter)

F. ROTATIONAL DYNAMICS

Newton's 2nd Law for Rotation

$$\tau_{net} = I\alpha$$

Work, Power & KE

$$W = \int \tau\, d\theta = \tau\Delta\theta \text{ (const } \tau)$$

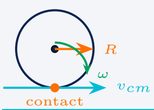
$$KE_{rot} = \frac{1}{2}I\omega^2; \quad P = \tau\omega$$

Work–Energy Theorem

$$W_{net} = \Delta KE = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2$$

G. ROLLING MOTION

Rolling Without Slipping



$$v_{cm} = R\omega; \quad a_{cm} = R\alpha$$

Contact point: $v = 0$; top point: $v = 2v_{cm}$

Total KE in Rolling

$$KE = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = \frac{1}{2}Mv_{cm}^2\left(1 + \frac{K^2}{R^2}\right)$$

Rolling on Incline (angle θ, length l)

$$a = \frac{g \sin\theta}{1 + K^2/R^2}$$

$$v = \sqrt{\frac{2gl\sin\theta}{1 + K^2/R^2}}; \quad t = \sqrt{\frac{2l(1 + K^2/R^2)}{g\sin\theta}}$$

K^2/R^2 Comparison Table

Body	K^2/R^2	Acc. rank
Solid sphere	2/5	1 (fastest)
Solid cylinder	1/2	2
Hollow sphere	2/3	3
Ring / Hollow cyl.	1	4 (slowest)

Trick

Lesser $K^2/R^2 \Rightarrow$ more acc. $\Rightarrow$ reaches bottom first.

H. JEE MAINS SHORTCUTS & TRICKS

1. COM of L-shaped/T-shaped rods:
Split into segments, use weighted average.

2. Removed-portion COM:

$$x_{cm} = \frac{Mx_M - mx_m}{M - m}$$

3. Angular impulse:
$L = \tau \cdot t$; $\quad \Delta L = \tau_{ext} \cdot \Delta t$

4. For a rolling body on incline:
Friction is static (no slipping), acts up the incline.
$f = \dfrac{MaK^2/R^2}{1 + K^2/R^2}\sin\theta$ (minimum static friction needed)

5. Rotation about different axes of disc:
Central axis: $\frac{MR^2}{2}$; Tangential: $\frac{3MR^2}{2}$
(use parallel axis theorem)

6. KE fraction in rolling:
$\dfrac{KE_{rot}}{KE_{total}} = \dfrac{K^2/R^2}{1 + K^2/R^2}$; $\dfrac{KE_{trans}}{KE_{total}} = \dfrac{1}{1 + K^2/R^2}$

7. Angular momentum of particle:
$L = mvr\sin\theta = mv \cdot d$ (d = perp. dist.)
For circular motion: $L = mvr$

8. Man–turntable:
$I_1\omega_1 = I_2\omega_2$; as $I \downarrow$, $\omega \uparrow$ (arms pulled in)

9. Condition for toppling vs slipping:
Toppling if $\mu > \tan\theta/2$ (for cube)
Slipping first if $\mu < \tan\theta$

Common Mistakes

- Applying perp. axis thm. to 3D bodies (only for lamina!)
- Forgetting $+Md^2$ in parallel axis theorem
- Using $v = r\omega$ when body is NOT pure rolling
- Taking COM = geometric center (only for uniform bodies)
- Ignoring friction in rolling dynamics (it provides torque!)

I. MASTER QUICK REVISION TABLE

Quantity	Formula
COM (2 particles)	$x_{cm} = \dfrac{m_1x_1 + m_2x_2}{m_1 + m_2}$
Velocity of COM	$v_{cm} = \dfrac{\sum m_i v_i}{M}$
Newton (system)	$\vec{F}_{ext} = M\vec{a}_{cm}$
Momentum consv.	$F_{ext} = 0 \Rightarrow p = \text{const}$
Coeff. restitution	$e = \dfrac{v_2 - v_1}{u_1 - u_2}$
Torque	$\tau = rF\sin\theta = I\alpha$
Angular momentum	$L = I\omega = mvr\sin\theta$
τ–L relation	$\tau = dL/dt$
Moment of inertia	$I = \sum m_i r_i^2 = MK^2$
Parallel axis	$I = I_{cm} + Md^2$
Perp. axis (lamina)	$I_z = I_x + I_y$
Rot. KE	$\frac{1}{2}I\omega^2$
Rolling KE	$\frac{1}{2}Mv^2(1 + K^2/R^2)$
Rolling acceleration	$a = \dfrac{g\sin\theta}{1 + K^2/R^2}$
Rolling condition	$v_{cm} = R\omega$
Cons. ang. mom.	$I_1\omega_1 = I_2\omega_2$